Definitions and key facts for section 4.2

The **null space** of an $m \times n$ matrix A, written as $\operatorname{Nul} A$, is the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation

$$\operatorname{Nul} A = \{\mathbf{x} \text{ in } \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Fact: For any $m \times n$ matrix A, $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$.

The **column space** of an $m \times n$ matrix A, written as $\operatorname{Col} A$, is set of all linear combinations of the columns of A. So if $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$, then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}.$$

In set notation,

$$\operatorname{Col} A = \{\mathbf{b} \text{ in } \mathbb{R}^m : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Fact: For any $m \times n$ matrix A, $\operatorname{Col} A$ is a subspace of $\mathbb{R}^m$.

A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

The **kernel** of a linear transformation T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector *in* W). The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

Fact: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A, then

$$\operatorname{Nul} A = \text{Kernel of } T \text{ and } \operatorname{Col} A = \text{Range of } T$$

as $T(\mathbf{x}) = A\mathbf{x}$ for all $x \in R^n$.